PMM U.S.S.R.,Vol.51,No.6,pp.723-730,1987
Printed in Great Britain

0021-8928/87 $10.00+0.00

PLANAR AND AXIALLY SYMMETRIC CONFIGURATIONS WHICH ARE CIRCUMVENTED WITH THE MAXIMUM CRITICAL MACH NUMBER*

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The structure of planar and axially symmetric configurations which, by satisfying a number of geometrical constraints, are circumvented in a boundless space or in a cylindrical channel by an ideal (non-viscous and non-thermally conducting) gas with a maximal critical Mach number M^* is found. The analysis is carried out using the "rectilinearity property" of a sonic line in "subsonic" flows (SF), the "principle of a maximum" for an SF and "comparison theorems" which are either taken from /1/ or serve as a generalization of the corresponding assertions from /1/. Following /1/, configurations are considered which have a plane or axis of symmetry parallel to the velocity $\mathbf{V}_\infty$ of the approach stream, while flows in which (including the boundary) the Mach number $M \leqslant 1$ are said to be "subsonic". As usual, by M^* we mean a value of M_∞ such that the inequality $M \leqslant 1$, which is satisfied in the whole stream when $M_\infty \leqslant M^*$, is violated when $M_\infty > M^*$.

The configurations investigated include closed bodies and the leading (trailing) parts of a semi-infinite plate or a circular cylinder in an unbounded flow and in a channel as well as lattices of symmetric profiles. Both in /1/, where the structure of closed planar and axially symmetric bodies was found, as well as in /2/, where such bodies were constructed numerically, the generatrices of all the configurations investigated contain the end planes or the segments replacing them of the maximum permissible slope (in modulus) and the "free" streamlines with $M \equiv 1$. Now, however, unlike in /1, 2/, segments of the horizontals are added to it in the general case. Furthermore, in the case of flows in channels and lattices, the configurations which have been found can be circumvented with the development of finite domains of advancing sonic flow.

1. Let us begin with the generatrix ab of a closed body in an unbounded flow (Fig.1,a) with the points a and b on the x-axis of a Cartesian (in the planar case) or cylindrical (in the axially symmetric case) coordinate system. The x-axis is directed along $\mathbf{V}_\infty$ and lies in the plane of symmetry of the body (or coincides with this axis) while the origin of coordinates and the linear scale are chosen such that $x_a = 0$ and $x_b = 1$. Here and subsequently, the indices $a, b, \ldots$ are ascribed to parameters at the corresponding points.

By virtue of the d'Alembert paradox which is valid in the case of a continuous subsonic flow, there is no difficulty in constructing, at any fixed $M_\infty < 1$, the set of bodies B with a coefficient of wave resistance c_x equal to zero. In particular, all sufficiently smooth thin bodies, the thickness of which, Y, is smaller, the closer M_∞ is to unity, will be members of this set. On the other hand, it is clear that, when there is a constraint on the smallest permissible size of the maximum midsection $Y \geqslant \delta$, on the volume of the body $\Omega \geqslant \Omega^m$ or on the area $S \geqslant S^m$ between the generatrix ab and the x-axis (in the planar case the volume per unit width of the body is identical with S), where δ, Ω^m and S^m are positive constants, a subsonic flow and, as a corollary, $c_x = 0$, are not realized at any $M_\infty < 1$. Furthermore, the introduction of an internal convex angularity of the contour ab (i.e. with an "external" angle greater than 2π) leads to the emergence, during its circumfluence, of a local supersonic zone (LSZ) and the appearance of wave resistance for any $M_\infty > 0$, no matter how small.

It is also necessary to take into account that the special techniques for the correction of profiles, which are being intensively developed and ensure a practically shock-free retardation of the gas in an LSZ and, consequently a value of c_x close to zero, assume a fixed circumfluence regime and do not guarantee $c_x \simeq 0$ even when M_∞ are smaller than the calculated value (but larger than M^*). There is, therefore, undoubtedly interest in configurations which are circumvented with the highest possible M^* but for which a subsonic flow with $c_x = 0$ is realized when $M_\infty \leqslant M^*$ according to the definition of M^* itself.

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*Prikl.Matem.Mekhan.,51,6,941-950,1987

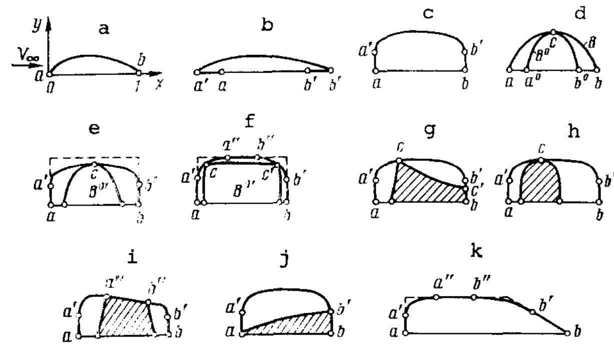

Fig.1

M^* is a functional of the shape of the body B which we shall write in the form $M^* = M^*(B)$ and, consequently, the problem of constructing the generatrix ab is a variational problem, the solution of which depends on additional constraints which, in particular, are of a geometrical nature. In the form being discussed this problem is non-standard. In spite of this, one may immediately point out its trivial solution which is important in its subsequent solution: at zero δ, Ω^m and S^m, the line segment ab realizes the maximum of $M^* = 1$, that is, the iptimal body B is a plate or a needle. Secondly, by carrying out a special "Gedanken experiment" which is characteristic in the case of variational problems in gas dynamics and invokes the result which has just been obtained (see /7, 8/), it is possible to establish the important elements of the solution. Apart from the constraint on the maximally permissible length taken for the linear scale, let just one of the conditions on Y, Ω or S figure in the problem as an additional condition. Forgetting the constraint on the length for a while, let us construct the optimal generatrix ab which, by joining the points a and b with $y_a = y_b = 0, x_a = 0$ and $x_b = 1$ and satisfying one of the conditions

$$Y \geqslant \delta,\ \Omega \geqslant \Omega^m,\ S \geqslant S^m \tag{1.1}$$

realizes a maximum in M^*. In an analogous formulation, which includes the segments aa' and bb' in the geneatrix ab as shown in Fig.1,b, it is possible by passing to the limit $x_{a'} \to -\infty$ and $x_{b'} \to \infty$ for any smooth convex arc $a'b'$ to approach as closely as may be desired to the plate or needle with the greatest possible $M^* = 1$. Since, however, the constraint on the length is not satisfied in the construction of such an "optimal" generatrix, we conclude from this that the optimal generatrix which satisfies the above-mentioned constraint will, in fact in the general case, contain the ends aa' and bb' as shown in Fig.1,c. When the maximum permissible length is specified, the same conclusion is also valid in the case of additional conditions differing from (1.1) if one applies similar reasoning to these as was done in the case of Fig.1,b. On the other hand, the conclusion, indicating the fundamental possibility of the ends appearing in the optimal generatrix in the general case, still does not guarantee their mandatory presence for any values of the decisive parameters of the problem. In this connection let us point to an analogy with the leading parts of fixed length which realize a minimum in c_x within the framework of Newton's law of resistance. Here, as is well-known /7-9/, axially symmetric leading parts always have bluntness but planar leading parts only have bluntness when $Y > 1$, where Y is referred to the length of the half thickness of the body. It therefore follows that the discourse surrounding Fig.1,b should only be considered as a guide. In order to draw more definite conclusions concerning the shape of the optimal generatrix ab we invoke an apparatus which is similar to that developed in /1/.

Let us begin with the rectilinearity property (RP) of a sonic line (SL) in a two-dimensional subsonic flow (SF). Such a line within a subsonic flow on which $M = 1$ while $M < 1$ on just one side of it is subsequently referred to as a sonic line. On the other side of the sonic line, $M \leqslant 1$. The case when $M = 1$ is necessary in the analysis of the circumfluence of lattices and bodies in a cylindrical channel. In spite of the latter refinement, we find by practically literally repeating the reasoning in /1/ that a sonic line in a subsonic flow is rectilinear, perpendicular to $\mathbf{V}$ at each of its points and cannot terminate within the flow. The impossibility of the occurrence of isolated "sonic points" within a subsonic flow which a subsonic flow which is proved using the same method may be looked upon at the same time as flowing from the rectilinearity property as a result of the absence of internal terminal points of a sonic line. It follows immediately from the RP that, during the subsonic circumfluence of closed planar (not necessarily symmetric) and axially symmetric bodies as well as the leading and trailing parts by an unbounded flow with $M_\infty < 1$, the Mach number can only attain its limiting value $M = 1$ on the surface of the body. The validity of this assertion, which is called following /1/ the "maximum principle" for a subsonic flow, is proved using the contrary assertion since, by assuming the opposite, we obtain a sonic line in the form of a half line (line) which goes away at infinity where, however, $M = M_\infty < 1$.

The "comparison theorem" (CT) which is proved by invoking the maximum principle (MP) and the essence of which is illustrated by Fig.1,d, serves as the next stage in the construction of a body around which flows occurs with maximum M^*. In fact, let a body B° lie within B while touching the latter at a point c which differs from a and b and let $M_\infty \leqslant M_\infty^\circ < 1$ and a "grade" be ascribed to the parameters of the flow circumventing B°. Next, let B and B° be surrounded by a subsonic flow without the occurrence of detachments (this is implied in /1/). Then, $V_c^\circ > V_c$.

Since the comparison theorem (CT) has been proved in /1/, we shall only dwell on points which refine this proof. The stress function ψ, corresponding to the body B, satisfies the equation

$$L(\psi) \equiv A_{11}\psi_{xx} - 2A_{12}\psi_{xy} + A_{22}\psi_{yy} + A_1\psi_x + A_2\psi_y = 0 \tag{1.2}$$

$$\begin{pmatrix} A_{11} = 1 - M_u^2, \quad A_{12} = M_u M_v, \quad A_{22} = 1 - M_v^2 \\ yA_1 = \nu A_{12}, \quad yA_2 = -\nu A_{22}, \quad M_u = u/a, \quad M_v = v/a \end{pmatrix}$$

Here, u and v are the x- and y-components of $\mathbf{V}$, a is the velocity of sound, and $\nu = 0$ and 1 respectively in the planar and axially symmetric cases. According to the definition of ψ

$$\rho u y^\nu = \psi_y, \quad \rho v y^\nu = -\psi_x \tag{1.3}$$

and, by virtue of the isoenergetic nature and isoentropic nature of the flow

$$2i + u^2 + v^2 = 2I_\infty, \quad s = s_\infty \tag{1.4}$$

where ρ is the density while i and s are the specific enthalpy and entropy. In this case it is convenient to consider the equations of state specified in the form

$$i = a^2\alpha(s, a^2), \quad \rho = \rho(s, a^2) \tag{1.5}$$

It is next assumed that all the quantities appearing in (1.2)-(1.5) are dimensionless with the critical velocity and density as the corresponding scales. Then, in the case of an ideal gas with constant heat capacities

$$2I_\infty = (\varkappa + 1)/(\varkappa - 1), \quad \alpha(s, a^2) = 1/(\varkappa - 1)$$

and, as a corollary of this, the coefficients of Eq.(1.2) are

$$A_{ij} = A_{ij}(\varkappa, y^\nu, \psi_x, \psi_y), \quad A_k = A_k(\varkappa, y^\nu, \psi_x, \psi_y) \tag{1.6}$$

where $\varkappa$ is the adiabatic index. In the case of gases with more complex thermodynamic arguments of the functions (1.6), the dimensionless values of the physical constants appearing in (1.5) are put in instead of $\varkappa$. Next, in the comparison of different flows, it is essential that A_{ij} and A_k on the one hand and A_{ij}° and A_k° on the other hand are identical functions of y^ν and ψ_x, ψ_y and ψ_x°, ψ_y° respectively. In the case of a non-ideal gas this is known to be valid when the dimensioned critical parameters of the flows being compared are identical. According to (1.6), this is not required in the case of an ideal gas. Furthermore, since, as is well-known /10/, ρ, a and $V = |\mathbf{V}|$ are two-valued functions of the flow density $j \equiv \rho V = \sqrt{\psi_x^2 + \psi_y^2}\, y^{-\nu/2}$, it is important in the subsequent development that the corresponding single-valued branches should exist when $M \leqslant 1$ and, consequently, also single-valued relationships between A_{ij} and A_k and $c\, y^\nu$, ψ_x and ψ_y.

The equations

$$\psi_\Gamma = 0, \quad \psi(x, y) \to y^{1+\nu} j_\infty/(1 + \nu) \quad \text{when} \quad r \equiv \sqrt{x^2 + y^2} \to \infty \tag{1.7}$$

serve as boundary conditions for the determination of ψ, where Γ is a flow line composed of two half paths ($x < x_a$ and $x > x_b$) of the x-axis and the generatrix of the body B. The equations and conditions (1.2)-(1.7) with the "grade" index accompanying all the symbols (apart from $x, y, \varkappa$ and ν) also determine the stream function ψ° corresponding to the circumfluence of B°. The function $\omega \equiv \psi^\circ - \psi$ is defined outside of B and on Γ. In the case of a continuous circumfluence, $\psi^\circ > 0$ everywhere outside of B°

$$\omega_\Gamma \geqslant 0, \quad \omega(x, y) \to y^\nu (j_\infty^\circ - j_\infty)/(1 + \nu) \geqslant 0 \quad \text{when} \quad r \to \infty \tag{1.8}$$

and, in the first condition, the equality only holds on the common segments of Γ and Γ°, including their tangent points, while, in the second condition, it only holds when $M_\infty^\circ = M_\infty$. In writing out the second condition, account has been taken of the fact that, if $0 \leqslant M \leqslant 1$, then j is a monotonically increasing function of M. Outside of B, the function ω satisfies the equation

$$L^\circ(\omega) + (A_{11}^\circ - A_{11})\psi_{xx} - 2(A_{12}^\circ - A_{12})\psi_{xy} + (A_{22}^\circ - A_{22})\psi_{yy} + (A_1^\circ - A_1)\psi_x + (A_2^\circ - A_2)\psi_y = 0 \tag{1.9}$$

in which the coefficients of the operator L° are not functions of ω_x and ω_y but the same functions of ψ_x° and ψ_y° as in $L^\circ(\psi^\circ)$. It follows from (1.9) that ω cannot have maxima or minima at internal points of a subsonic flow which differ from an infinitely remote point. Actually, by assuming the opposite to be the case, we obtain that, at such points,

$$\omega_x \equiv \psi_x^\circ - \psi_x = 0, \quad \omega_y \equiv \psi_y^\circ - \psi_y = 0$$

By virtue of these equalities and the fact that, according to what has previously been said, the coefficients $A_{ij}, A_k, A_{ij}^\circ$ and A_k° in a subsonic flow are single-valued functions of ψ_x and ψ_y or ψ_x° and ψ_y°, here (1.9) reduces to the equation

$$(1-M_u^2)\,\omega_{xx} - 2M_uM_v\omega_{xy} + (1-M_v^2)\,\omega_{yy} = 0 \tag{1.10}$$

which, as is well-known, cannot be satisfied, when $M^2 \equiv M_u^2 + M_v^2 \leqslant 1$, either at maximum or minimum points of ω. Next, let n be measured off along the normal to B or to B° at the point c. Then, allowing for the result which has been obtained and the fact that, by virtue of (1.3), $\partial\psi/\partial n \equiv \psi_n = \rho V y^\nu$, we shall have

$$(\partial\omega/\partial n)_c \equiv y_c^\nu\,(j^\circ - j)_c \geqslant 0$$

Since, when $M \leqslant 1$, the density of the flow increases montonically with V, it follows from this that $V_c^\circ \geqslant V_c$. Actually, however, the strict inequality

$$V_c^\circ > V_c \tag{1.11}$$

is satisfied at all points of contact of the bodies B and $B^\circ \in B$ which do not coincide identically. The transition to (1.1), which completes the proof of the comparison theorem referred to in /1/ as the "boundary point lemma", was carried out there using an exceedingly complex method and was bounded by the "regular points" (RP) of B at which the second derivatives of ψ with respect to x and y are finite. In practice, (1.11) follows almost directly from (1.9) and in a more general case. In fact, let us assume that $V_c^\circ = V_c$. Then, as at the points of the extremum of ω, here $A_{ij}^\circ = A_{ij}$ and $A_k^\circ = A_k$ and, what is more, $\omega_n = \omega_\tau = 0$, where τ is read off along the generatrix ab and, consequently, $\omega_x = \omega_y = 0$. Hence, Eq.(1.9) in a small neighbourhood of c is found to be as close to (1.10) as may be desired and not only when c is a regular point but also for other non-regular points (but they are only subsequently encountered) in the approximation to which the second derivatives of ψ tend to infinity more slowly than the coefficients in (1.9), standing in front of them, tend to zero. At the same time, by virtue of the maximum principle in the case of ω, the "half of the minimum" of the surface $\omega = \omega(x, y)$ which is cut off by the generatrix of B is adjoined to the point c when $\omega_\tau = \omega_n = 0$. This, however, is impossible as in such a case the left side of (1.10) would differ from zero by a finite amount when $M \leqslant 1$ and, consequently, Eq.(1.9) could not be satisfied in a small neighbourhood of c. Finally, we note that, in (1.9), L° and ψ can be simultaneously replaced by L and ψ°. When this is done, the conditions formulated above on $\psi_{xx}, \ldots$ are replaced by the analogous conditions on $\psi_{xx}^\circ, \ldots$.

Using the comparison theorem, let us now construct a closed body of fixed length which, by satisfying the three conditions

$$y \leqslant \Delta,\ \Omega \geqslant \Omega^m,\ S \geqslant S^m \tag{1.12}$$

realizes a maximum in M^*. The main difference in the given formulation from the generalization of the formulation of the variational problem from /1/, to which the inequalities (1.1) correspond instead of the equalities in /1/, is not in the number of constraints but in the sense of the first of them. In (1.12), the thickness of the body, when Ω^m and S^m differ from zero, must not exceed the maximum permissible value of Δ while, in /1/ and in (1.1), on the other hand, the maximum midsection of the body cannot be less than the minimum permissible value of δ. By virtue of (1.12), the generatrix of the body does not pass outside of the rectangle B_Δ: $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant \Delta$ which is shown in Fig.1,e and f by the broken lines. If Ω_Δ and S_Δ are the values of Ω and S corresponding to it, then $\Omega^m \leqslant \Omega_\Delta$ and $S^m \leqslant S_\Delta$, and, moreover, when $\Omega^m = \Omega_\Delta$ or $S^m = S_\Delta$, the generatrix ab has breaks and $M^* = 0$.

Let us show that the contour of the closed optimal body B of fixed length which satisfies conditions (1.12) and which realizes a maximum value of M^* in an unbounded subsonic flow depending on the values of Δ, Ω^m and S^m either, as in /1/, consists of the ends $x = 0$ and $x = 1$ and the flow line $M = 1$ joining them, or it contains an additional segment of the horizontal $y = \Delta$. Both situations are depicted in Fig.1,e and f, where $M = 1$ on $a'b'$ and $a'b''$ and $b''b'$ respectively.

According to the equations of motion, $\partial p/\partial n = -\rho V^2 \partial\vartheta/\partial\tau$ where ϑ is the angle between $\mathbf{V}$ and the x-axis and p is the pressure. It follows from this that, in a subsonic flow in which we have $\partial\vartheta/\partial\tau \leqslant 0$ on the segments of the "sonic" streamlines where $\partial p/\partial n \geqslant 0$, these segments and, with them, also the optimal contours in Fig.1,e and f are consequently non-concave. In such a case any body B°, which satisfies the conditions listed above, is either identical to B or it just partially emerges out of it, lying nevertheless as B within B_Δ. Let us surmise that it is B° rather than B which is optimal, i.e. $M^{*\circ} > M^*$. Then, after an affine contraction which does not change the cirumfluence, we arrive at a body $B^{\circ\prime}$ which comes into contact with B at not less than one point (c or c and c') of the interval $M = V = 1$. However, according to the comparison theorem, $V^\circ > V = 1$ at such points and, consequently, contrary to the postulate, the flow around B° is not subsonic and B°, by virtue of the definition, is non-optimal.

Within the framework of the comparison theorem the result in /1/ is readily generalized to bodies which satisfy, for the same maximum permissible length, not just one of the conditions (1.1) but all three at once. In this case the thickest of the three bodies with a single sonic flow line (Fig.1,e), constructed respectively for $Y = \delta$, $\Omega = \Omega^m$ and $S = S^m$, is optimal. Another generalization is obtained if, instead of the first condition from (1.1), the more natural requirement of the disposition of a specified body G within B is introduced. Examples of optimal configurations which are obtained when this is done with the help of the comparison theorem in the case when the position of G is fixed relative to the interval ab of the x-axis and there are no conditions on Ω and S are shown in Fig.1,g to j, where the bodies G are hatched in and $M = 1$ on $a'cb'$, $a'a''$, $b''b'$ and $a'b'$. At the same time, the contours of B and G can have both individual common points (c in Fig.1,g and h) as well as common segments ($c'b$ in Fig.1,g, $a''b''$ in Fig.1, i and $b'b$ in Fig.1,j). If a displacement of a body G is permitted, then its position is taken such that, for any permissible shifts of G, the thickness of B in the neighbourhood of just one of the intervals $M = 1$ increases. In particular, bodies G with a vertical plane of symmetry must be located midway along the interval ab.

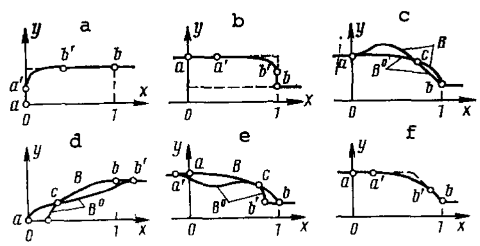

Fig.2

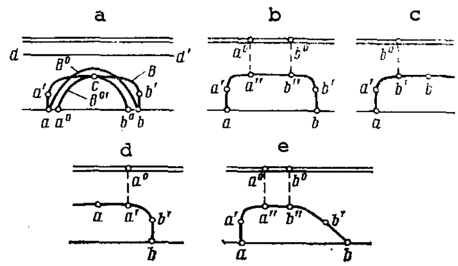

Fig.3

2. In addition to the contours of closed bodies there is also interest in "unclosed" generatrices which may be considered as the leading or trailing parts of sufficiently long bodies and, in the axially symmetric case, as the transition sections linking cylinders of different radii. At the same time it is natural to replace long bodies by a semi-infinite plate or a circular cylinder with generatrices parallel to $\mathbf{V}_\infty$.

Schemes for the configurations being considered are shown in Fig.2,a and b which also illustrate the choice of coordinate axes and linear scale. Unlike in Fig.1, it is now just one of the end points of the required optimal contour ab which does not lie on the x-axis. In the case of the transitional axially symmetric segments (Fig.2,b) the ordinates of both terminal points differ from zero. Apart from the maximum permissible length of the body which, as previously, is taken as the linear scale, the maximum and minimum permissible values of y on ab are fixed as a result of which the required generatrices must be located in rectangles which are bounded by intervals of the x- and y-axes and the dashed lines in Fig. 2,a and b.

As previously, the conditions on Ω and S from (1.1) may be introduced as additional constraints and, here, by Ω and S, we understand the volumes and areas of the above-mentioned rectangles or bodies, which are obtained by rotating them around the x-axis, cut off by the generatrix ab. The forces which act on the leading and trailing parts and on the transitional segments are obviously different from zero at any Mach numbers. Nevertheless, in this case also, if in the whole of the flow $M \leqslant 1$ when there is continuous circumfluence, the coefficient of wave resistance is equal to zero as in the case of closed bodies. Here, there is also interest in generatrices which realize a maximum value of M^*.

The construction of such optimal generatrices, as in the case of bodies with a closed contour is carried out with the aid of a version of the comparison theorem the formulation and proof of which barely differ from those for the comparison theorem from /1/ which was used above. The principal difference lies in the fact that the condition as $r \to \infty$ from (1.7) is now replaced by

$$\psi(x, y) \to j_\infty \Psi(x/\Delta,\ y/\Delta,\ \delta/\Delta,\ \nu) \text{ when } r/\Delta \to \infty$$

Here Δ and δ are the maximum and minimum permissible values of y on the contour

($\delta = y_a = 0$, $\Delta = y_b$ in Fig.2,a and $\delta = y_b$, $\Delta = y_a$ in Fig.2,b) and the non-negative function Ψ is independent of the shape of the leading and trailing parts or the transitional segment and $\Psi(-\infty, 1, \ldots) = \Psi(\infty, \delta/\Delta, \ldots) = 0$. Furthermore, we shall subsequently consider bodies with identical Δ and δ, i.e. which solely differ in the finite interval of the x-axis (as applied to the leading parts and the corresponding transition segments when $0 < x < 2$ and to their trailing analogues when $-1 < x < 1$). Allowing for what has been said and Figs.2,c to e which explain the meaning of the comparison theorem which is subsequently used, we conclude that, in the case of bodies B and $B° \Subset B$ when $1 > M_\infty° \geqslant M_\infty$, the inequality $V° > V$ holds at points of contact and on common segments of the contours of B and $B°$. An obvious exception is consitituted by infinitely remote points of the halflines $y = \delta$ and $y = \Delta$, where $M_\infty° = M_\infty$ and the velocities of the flows being compared are identical.

With the help of the comparison theorem it is now possible to establish that, in the cases being considered, the optimal generatrices consists of the ends ($x = 0$ for the leading parts and $x = 1$ for the trailing parts and the corresponding transition parts), the streamlines $M = 1$ and segments of the horizontals $y = \Delta$, as can be seen from Figs.2,a and b and, moreover, the segments of the horizontals, only appear when there are constraints on Ω and S. As Ω^m and S^m decrease, which cooresponds to an "easing off" of the constraints, M^* increases, attaining a maximum in the case of the generatrix ab without a horizontal part when Δ and δ are fixed ($\delta = 0$ in the planar case). The same generatrix is optimal when none of the conditions (1.12) apply, i.e. also among configurations with $y > \Delta$.

The validity of everything that has been said is proved by assuming that the opposite is true. In doing this, as a rule, instead of the affine contraction employed in Sect.1, it is sufficient to shift the body $B°$ being compared to the right (in the case of the leading parts) or to the left (in the case of the trailing parts) until one of the situations depicted in Figs.2,c and d inevitably arises, leading to a contradiction. An affine contraction is only invoked in the comparison of the optimal body with thicker bodies ($y > \Delta$). Finally, we note that, by virtue of what has been said, the sharpened leading parts constructed in /11/ cannot be optimal (with respect to M^*).

3. Now let a body B with a non-concave generatrix be surrounded by a subsonic flow with $M \equiv M_\infty \leqslant 1$ as $|x| \to \infty$, but not in an unbounded space but rather in a cylindrical channel (Fig.3,a where the double line is the generatrix of the wall of the channel). Let another body $B°$ of the same length also be surrounded by a subsonic flow with $M_\infty° \geqslant M_\infty$ and let the generatrices of B and $B°$ intersect. By mentally contracting $B°$ together with the tube in an affine manner with the centre on the x-axis within $B°$ and, consequently, not changing the flow, we arrive at a body $B^{\circ\prime} \Subset B$ which is such that the generatrices of B and $B^{\circ\prime}$ will either touch one another or have common segments. Then, $V^{\circ\prime} > V$ at common points of B and $B^{\circ\prime}$ which are different from a and b which consitutes a further version of the comparison theorem.

In proving this, we must take account, in addition to what has been said previously, of the fact that the flow lines of the subsonic flow flowing round B initially become monotonically more remote from the axis of the channel and then again monotonically approach it (if the body B has a plane of symmetry normal to the x-axis, this follows at once from the maximum principle for v). Hence, on the line dd', into which the wall of the tube is transformed under the affine contraction, $\psi \leqslant \psi(-\infty, y_d)$. Consequently, the function $\omega(x, y) \equiv \psi^{\circ\prime}(x, y) - \psi(x, y)$, which is defined in a band bounded by the axis of the channel, the generatrix ab and the straight line dd', is non-negative on the whole of the boundary which has been indicated. The further course of the proof of the comparison theorem is the same as that presented in /1/ with the refinements of Sect.1.

In addition to the comparison theorem, let us further recall the property of the rectilinearity of a sonic line in a subsonic flow which was invoked in Sect.1. Now, however, the appearance of straight sonic lines with $M < 1$ on the one hand and $M \equiv 1$, on the other hand, in a subsonic flow does not contradict this property. Such sonic lines bound the "sonic regions" (SR), i.e. domains of uniform sonic flow.

The version of the comparison theorem which has been formulated and yet another version which is similar to it for bodies with an open contour together with what has been said above in connection with sonic lines and sonic regions in a subsonic flow enables one to find the structure of the closed and unclosed generatrices of such bodies of fixed maximum permissible length which, when subject to the additional constraints (1.1), (1.12) or parts of them (for example, without the conditions on Y and y) realize a maximum value of M^*. When this is done, together with configurations of the same structure as in the case of an unbounded flow (they are knowingly realized in the case of a small "loading" of the channel, when $k_\Omega \equiv \Omega^m/R^{2+\nu} \ll 1$ and $k_S \equiv S^m/R^2 \ll 1$, where R is the half-height or the radius of the channel), horizontal segments appear, starting from certain values of k_Ω and k_S even in the absence of the condition on y from (1.12) for the optimal generatrices. The sonic domains are located over such segments. These domains are bounded from two sides or from one side by rectilinear sonic lines which are normal to the walls of the channel.

Figs.3,b-d illustrate what has been said and, in Fig.3,b, $M = 1$ on $a'b'$ and in the rectangle $a''a^\circ b^\circ b''$, in Fig.3,c, on $a'b'$ and everywhere to the right of $b'b^\circ$ and, in Fig.3,d, on $a'b'$ and everywhere to the left of $a'a^\circ$. The latter example is also of interest in that $M^* = M_\infty = 1$ for this case. In an unbounded flow this only holds in trivial examples of the flow round a plate or needle of zero thickness. We note, by the way, that, in spite of the absence of discontinuities and the increase in entropy associated with them, c_x, which is defined in the usual manner, differs from zero in the cases of Figs.3,c and d (moreover, $c_x < 0$ in the case of Fig.3,d). It is natural that, since the realization of the flows depicted in Figs.3,b-d requires the maintenance of special boundary conditions as $|x| \to \infty$, they are only of theoretical interest.

From the configurations which realize a maximum value of M^* during circumfluence in a planar channel, it is possible to construct lattices of symmetric profiles with fronts normal to $\mathbf{V}_\infty$. Although such lattices are of no interest in applications, the peculiarities inherent in them may also turn out to be characteristic of lattices made up from "supporting" profiles, at least, at a low level of the corresponding forces and moments.

In concluding, we shall make several remarks of a general nature.

First, the flow around all of the bodies which have been considered in a viscous gas, apart from the leading parts, will be accompanied by the occurrence of developed detached zones around the rear ends which changes the whole picture of the flow. Therefore, even in the formulation of the problem, it is advisable to introduce constraints which ensure the wider applicability of the results obtained within the framework of an ideal gas. Following /1/, this is most simply done by requiring that the angle of slope of the generatrix of the body $\vartheta_w(x)$ to the x-axis should satisfy the inequality

$$\vartheta_w(x) \geqslant \vartheta_0 > -\pi/2 \tag{3.1}$$

with a specified constant ϑ_0. The introduction of condition (3.1) leads to the replacement of the rear ends by the intervals of the straight lines $\vartheta_w(x) \equiv \vartheta_0$ as shown in Fig.1,k, Fig.2,f and Fig.3,e. Of course, when $|\vartheta_0| < \pi/2$, a discontinuity arises in the neighbourhood of the retardation point b. However, at large Reynolds numbers on passing from the end to the tip, the detached zone and, together with it, the effect of viscosity on the flow as a whole decreases rapidly. A further detachment may be located in the neighbourhood of the point b' to the right of which $(p_\tau)_{b'} \equiv (\partial p/\partial\tau)_{b'} = \infty$ outside of the dependence on the magnitude of ϑ_0. In order to get rid of this it is necessary, in addition to (3.1), to introduce a constraint on p_τ in the form $p_\tau \leqslant \xi$ with a specified positive constant (or function) ξ and, more precisely, on a certain dimensionless combination (the "detachment criterion") which is proportional to p_τ. When this is done, the segment $p_\tau \equiv \xi$ appears in the neighbourhood of b' in the case of the optimal contour.

Secondly, in the example which have been considered, the comparison theorems have enabled us to find contours which realize a global maximum for M^* and it turned out that all the solutions constructed only consist of segments of a boundary extremum either along directions (the ends, intervals of the straight lines $y = \Delta$ and $\vartheta_w = \vartheta_0$) or along a phase coordinate (the flow line $M = 1$). On the other hand, within the framework of the traditional approaches to the theory of optimal control or variational calculus in gas dynamics, it is only possible to satisfy the necessary conditions for a local optimum, and the question regarding the construction of all possible optimal solutions and the choice of the best of them, i.e. regarding the "synthesis of the optimal control", usually remains open.

Unfortunately, this advantage of comparison theorems is compensated by the narrowness of the range of problems to which they can be applied. For instance, the lack of the corresponding comparison theorem precludes any effecient solution of the problem analogous to those considered, for example, in the case of a supporting profile. Since the problems which have been solved can be used to work out more-general approaches, the need arises for them to be reformulated in the traditional form of variational problems in gas dynamics. The new formulation reduces the treatment of the problem to the construction of the contours of bodies of fixed maximum permissible length which realize the maximum of S and Ω during subsonic $(M \leqslant 1)$ circumfluence with a specified M_∞. Without dwelling on the problems associated with the use of direct and indirect approaches to the solution of such problems, we shall merely mention one instance which arises in work using the method of Lagrange multipliers which is widely employed in supersonic gas dynamics /7/. When $|\vartheta_0| < \pi/2$ at the rear retardation point (Fig.1,k, Fig.2,f and Fig.3,e) certain derivatives of the flow parameters are infinite. Hence, if the point b is displayed during the variation of the contour, it may lead to a breakdown in the assumption that all the variations are small which is used in the method of Lagrangian multipliers.

The simplest way to avoid such trouble lies in combining the origin of the coordinate system not with the starting cross-section of the body but with the terminal cross-section. When this is done, the length of the body is reduced due to a variation in x_a for fixed $x_b \equiv 0$. By the way, the absence in the case of the contours which have been found of segments of a bilateral extremum indicate the strong dependence of M^* on their shape since a deformation δy of the ordinate of the contour on an interval Δx reduces M^* by $O(\delta y \cdot \Delta x)$. Deformation of the same segment of a bilateral extremum would only change M^* by $O[(\delta y)^2 \Delta x]$.

Finally, one last point. It is impossible not to see a number of similar features in the problems of hydro- and gas-dynamics which have been considered here and have previously been solved. Apart from the end planes which are characteristic, as has already been mentioned, in the case of the leading parts of minimal resistance at supersonic and hypersonic velocities, there are, above all, the free lines of flow, where $V = \text{const}$. It is interesting that the configurations formed by such lines (and again by the ends) ensure minimum resistance during cavitational circumfluence by an incompressible fluid /12/ and, moreover, the analysis in the latter paper was also based on the invocation of the corresponding comparison theorems. As far as the straight sonic lines and the sonic zones adjoining them are concerned, these elements are typical in the case of a subsonic flow in dimains with $M = 1$ on one of the flow boundaries /13/.

The author thanks V.A. Vostretsov for helping with this research.

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Translated by E.L.S.